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COMMENT

A theorem for a class of motions in a spherical geometry

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Abstract. Considering the motion, at classical level, of a point particle constrained to move under the influence of a conservative centre of force, on a N-sphere $S^{(N)}$, embedded in a Euclidean $(N+1)$-dimensional space, we have shown that, if the motion on $S^{(N)}$ is obtained from the motion on a tangent N-plane $\Pi^{(N)}$ by means of a central projection, then the equations of motion on $S^{(N)}$ can be obtained from those on the Euclidean $\Pi^{(N)}$ by a local reparametrisation of time. Some consequences of the theorem are also discussed.

The non-relativistic motion of a particle on a fixed N-sphere $S^{(N)}$ (embedded in Euclidean $(N+1)$-dimensional space) in which a conservative centre of force is defined has been investigated some time ago by Higgs (1979) and Leemon (1979). In Higgs' paper, the motion on $S^{(N)}$ was defined by means of a central (also known as gnomonic) projection from the motion on a N-plane $\Pi^{(N)}$, tangent to $S^{(N)}$ at a given point. At a classical level, he then proved the beautiful result that the dynamical symmetries in $S^{(N)}$ are the same as those existing in $\Pi^{(N)}$, a property that is also valid quantum mechanically.

In this paper we prove a theorem which further clarifies the above problem, namely: if the (classical) motion on $S^{(N)}$ is obtained from a central motion on $\Pi^{(N)}$ by means of a central or gnomic projection, the equations of motion on $S^{(N)}$ can be derived from the equations of motion on the Euclidean N-plane $\Pi^{(N)}$ by a local reparametrisation of time.

Before proving the theorem we briefly derive, under the assumption of central projection, the metric of $S^{(N)}$, in terms of the Cartesian coordinates on the tangent N-plane $\Pi^{(N)}$. Let R be the radius of $S^{(N)}$, supposed to be fixed. Denoting by $X_1, X_2, \ldots, X_{N+1}$ the coordinates of a generic point $M \in S^{(N)}$, with respect to a coordinate system whose origin coincides with the centre of $S^{(N)}$, one has

$$X_1^2 + X_2^2 + \ldots + X_{N+1}^2 = R^2 = \lambda^{-1}, \tag{1}$$

where λ is the curvature. Let $\Pi^{(N)}$ be the tangent N-plane to $S^{(N)}$ at the point $(0, 0, \ldots, -R)$ and M' a generic point of $\Pi^{(N)}$ whose coordinates will be denoted by $(x_1, x_2, \ldots, x_N, -R)$. The equation of the straight line passing through M and M' will be

$$\frac{\xi_1 - X_1}{X_1 - x_1} = \frac{\xi_2 - X_2}{X_2 - x_2} = \ldots = \frac{\xi_{N+1} - X_{N+1}}{X_{N+1} + R}. \tag{2}$$

Since M and M' are related by a central projection, one must impose the condition

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that (2) passes through the centre of $S^{(N)}$. Consequently,

$$X_i = kx_i \qquad (i = 1, 2, \ldots, N), \qquad X_{N+1} = -kR, \tag{3}$$

where k can be calculated by substituting (4) into (1):

$$k = \pm(1 + \lambda r^2)^{-1/2}. \tag{4}$$

In (4), r denotes the Euclidean distance in $\Pi^{(N)}$:

$$r = (x_1^2 + x_2^2 + \ldots + x_N^2)^{1/2}. \tag{5}$$

Therefore, one obtains

$$\begin{aligned} X_i &= \pm(1 + \lambda r^2)^{-1/2} x_i \qquad (i = 1, 2, \ldots, N), \\ X_{N+1} &= \mp(1 + \lambda r^2)^{-1/2} R. \end{aligned} \tag{6}$$

The double signs in (6) reflect the fact that the correspondence implied by the central projection is not one-to-one. The metric on $S^{(N)}$

$$\mathrm{d}s^2 = \mathrm{d}X_1^2 + \mathrm{d}X_2^2 + \ldots \mathrm{d}X_{N+1}^2$$

can be easily calculated from (6). One gets

$$\mathrm{d}s^2 = (1 + \lambda r^2)^{-1}\,\mathrm{d}\boldsymbol{x}^2 - \lambda(1 + \lambda r^2)^{-2}(\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x})^2, \tag{7}$$

where $\boldsymbol{x} \equiv (x_1, x_2, \ldots, x_N)$.

Assuming that the particle has unit mass and that the potential is of the form $V(r)$, with r defined by (5), one gets from (7) the Lagrangian in $S^{(N)}$

$$\mathscr{L}_\tau = \frac{1}{2}\frac{1}{1 + \lambda r^2}\left(\dot{\boldsymbol{x}}_\tau^2 - \lambda\frac{(\boldsymbol{x} \cdot \dot{\boldsymbol{x}}_\tau)^2}{1 + \lambda r^2}\right) - V(r), \tag{8}$$

where $\dot{\boldsymbol{x}}_\tau = \mathrm{d}\boldsymbol{x}/\mathrm{d}\tau$, τ being the time variable on $S^{(N)}$. Notice that the motion on $\Pi^{(N)}$ will be parametrised by t and we shall write $\dot{\boldsymbol{x}}_t = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$ for the corresponding derivative.

Expression (8) is entirely equivalent to that obtained by Higgs. The Euler–Lagrange equations of motion corresponding to it may be shown to be given by

$$\ddot{\boldsymbol{x}}_\tau - 2\lambda\frac{\boldsymbol{x} \cdot \dot{\boldsymbol{x}}_\tau}{1 + \lambda r^2}\dot{\boldsymbol{x}}_\tau + \frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}(1 + \lambda r^2)^2\boldsymbol{x} = 0, \tag{9}$$

which define a system of N ordinary nonlinear differential equations. Of course, the nonlinear terms in (9) vanish when the curvature λ goes to zero.

We now consider a central motion on the tangent N-plane $\Pi^{(N)}$, whose Lagrangian, in terms of the time variable t, is

$$\mathscr{L}_t = \tfrac{1}{2}\dot{\boldsymbol{x}}_t^2 - V(r). \tag{10}$$

The corresponding Euler–Lagrange equations are, of course,

$$\ddot{\boldsymbol{x}}_t + \frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}\boldsymbol{x} = 0. \tag{11}$$

Assume now that between the differentials of the time variables t and τ the following relation holds:

$$\mathrm{d}t = (1 + \lambda r^2)\,\mathrm{d}\tau. \tag{12}$$

It is reasonable to say that equation (12) is local since its RHS is a function of r.

The following equations are simple consequences of (12):

$$\dot{\boldsymbol{x}}_t = \dot{\boldsymbol{x}}_\tau/(1+\lambda r^2), \tag{13a}$$

$$\ddot{\boldsymbol{x}}_t = (1+\lambda r^2)^{-2}\{\ddot{\boldsymbol{x}}_\tau - 2\lambda[(\boldsymbol{x}\cdot\dot{\boldsymbol{x}}_\tau)/(1+\lambda r^2)]\dot{\boldsymbol{x}}_\tau\} \tag{13b}$$

By substituting (13*b*) into (11) one readily gets

$$\ddot{\boldsymbol{x}}_\tau - 2\lambda\frac{\boldsymbol{x}\cdot\dot{\boldsymbol{x}}_\tau}{1+\lambda r^2}\dot{\boldsymbol{x}}_\tau + \frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}(1+\lambda r^2)^2\boldsymbol{x} = 0,$$

a result which coincides with equation (9). This completes the proof of our theorem. We now discuss some of the consequences of it. Firstly, since the orbits are obtained from the equations of motion by eliminating the time, it follows that, for a given potential, they will obey similar differential equations, for both $S^{(N)}$ and $\Pi^{(N)}$ motions. In particular, closed orbits in $\Pi^{(N)}$ will correspond to closed orbits in $S^{(N)}$.

Secondly, solutions to the nonlinear system (9) can be obtained from solutions to the linear system (11) by integrating (13*a*). This is particularly easy to obtain in the case of $S^{(1)}$ by projecting simple motions in $\Pi^{(1)}$ such as the uniform ($V=0$) or harmonic motion ($V=\frac{1}{2}\omega^2x^2$).

Thirdly, one already sees that the theorem strongly indicates that the dynamical symmetries are the same both for $S^{(N)}$ and $\Pi^{(N)}$. That this is so has been proved by Higgs using the cannonical formalism and investigating the Poisson bracket algebras of the constants of motion. However, we wish to point out that, in the present context, it is a simple matter to associate to a given constant of motion in $\Pi^{(N)}$ a corresponding constant motion in $S^{(N)}$. We will illustrate this for the more motivated case of the three-sphere $S^{(3)}$. Any central motion in $\Pi^{(3)}$ admits the angular momentum vector $\boldsymbol{L}_t$ as a constant of motion:

$$\boldsymbol{L}_t = \boldsymbol{x}\times\dot{\boldsymbol{x}}_t, \qquad \dot{\boldsymbol{L}}_t = 0. \tag{14}$$

With the help of (13*a*), we define the angular momentum on $S^{(3)}$ as

$$\boldsymbol{L}_\tau = (\boldsymbol{x}\times\dot{\boldsymbol{x}}_\tau)/(1+\lambda r^2) \tag{15}$$

which obeys the conservation law

$$\dot{\boldsymbol{L}}_\tau = 0 \tag{16}$$

as can be easily checked using (9).

In much the same way, it can be easily shown that, for the Kepler (or Coulomb attractive) problem

$$V_{\mathrm{K}} = -\mu/r \qquad (\mu>0) \tag{17}$$

the following vector is a constant of motion on $S^{(3)}$

$$\boldsymbol{M}_\tau = \boldsymbol{L}_\tau\times[\dot{\boldsymbol{x}}_\tau/(1+\lambda r^2)] + \mu\boldsymbol{x}/r, \tag{18}$$

that is,

$$\dot{\boldsymbol{M}}_\tau = 0. \tag{19}$$

Equation (18) corresponds, for $\lambda=0$, to the well known Laplace–Runge–Lenz vector constant of motion in $\Pi^{(3)}$ (Laplace 1799, Runge 1919, Lenz 1924). They are, of course, only six independent constants of motion in view of the identity

$$\boldsymbol{M}_\tau^2 = \mu^2 + 2E\boldsymbol{L}_\tau^2 + \lambda(\boldsymbol{L}_\tau^2)^2 \tag{20}$$

where E is the energy

$$E = \tfrac{1}{2}[(1+2r^2)^{-1}\dot{x}_\tau^2 - \lambda(x\cdot\dot{x}_\tau)^2/(1+\lambda r^2)^2] + V_K(r). \tag{21}$$

Similarly, for the isotropic harmonic oscillator potential,

$$V_{HO} = \tfrac{1}{2}\omega^2 r^2, \tag{22}$$

the components of the (generalised) Fradkin tensor (Fradkin 1967)

$$(F_{ij})_\tau = \frac{1}{2}\left(\frac{(\dot{x}_i)_\tau}{1+\lambda r^2}\,\frac{(\dot{x}_j)_\tau}{1+\lambda r^2} + \omega^2(x_i x_j)(\tau)\right) \tag{23}$$

are constants of motion on $S^{(3)}$, that is

$$(\dot{F}_{ij})_\tau = 0 \tag{24}$$

where the dot implies τ time differentiation.

Finally, a few words about the physical meaning of the potentials (17) and (22) as defining conservative centres of force in $S^{(3)}$. By using hyperspherical polar coordinates in $r^{(4)}$, namely

$$(X_1, X_2, X_3, X_4) \equiv R(\sin\alpha\,\sin\theta\,\cos\varphi, \sin\alpha\,\sin\theta\,\sin\varphi, \sin\alpha\,\cos\theta, \cos\alpha)$$

$$(0 \leqslant \alpha, \theta \leqslant \pi, 0 \leqslant \varphi \leqslant 2\pi)$$

one gets, from (3), that

$$\sum_{i=1}^{3} X_i^2 = r^2/(1+\lambda r^2) = \lambda^{-1}\sin^2\alpha.$$

Hence, $\lambda^{1/2} r = \tan\alpha$. Using this result, one obtains from (17) and (22) that

$$V_K = -\mu\lambda^{1/2}\cot\alpha, \tag{25}$$

$$V_{H0} = \tfrac{1}{2}\omega^2\lambda^{-1}\tan^2\alpha. \tag{26}$$

As pointed out by Higgs, in the form (25) the Kepler potential ($\mu > 0$) has an attractive singularity at the north pole ($\alpha = 0$) and an equal repulsive singularity at the south pole ($\alpha = \Pi$). Notice that it vanishes at the equator ($\alpha = \Pi/2$). On the other hand, for the isotropic harmonic oscillator potential (26) we have two attractive centres, one in each pole. However, the equator acts as a reflecting barrier since the potential is singular for $\alpha = \Pi/2$. Therefore, the motion will be confined to a hemisphere.

We conclude with a few remarks. The introduction of conservative centres of force in a spherical geometry can be conceived in a number of ways. However, the formulation which makes use of central projection presents features not shared by other formulations such as those based on the orthogonal projection (Lakshmanan and Eswaran 1975) or the stereographic one. Among those features stands the theorem we have proved in this note, which is based on a reparametrisation of time. In view of the absolute role of time in classical dynamics, the introduction of different times is, clearly, a mathematical device, whose main interest, in this context, seems to reside in the theorem we have proved above.

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